

**Formula Sheet** (Rebecca Turner, 2019-11-12)

$$\mathbf{a} \cdot \mathbf{b} = \sum_i a_i b_i = |\mathbf{a}| |\mathbf{b}| \cos \theta.$$

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle. \quad |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \theta.$$

Param. eqns. of line through  $\langle x_0, y_0, z_0 \rangle$  par. to  $\langle a, b, c \rangle$ :

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

$$\text{Symm. eqns.: } \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Vec. eqn. of plane through  $\mathbf{r}$  with  $\mathbf{n}$  normal:

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0, \quad \mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0.$$

Length along a vec. fn.  $\mathbf{r}(t)$ :

$$\int_a^b |\mathbf{r}'(t)| dt = \int_a^b \sqrt{\sum_i r_i'(t)^2} dt,$$

Unit tang.  $\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)|$ , so curvature of  $\mathbf{r}(t)$  w/r/t the arc len. fn.  $s$ :

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

$$\text{Unit normal: } \mathbf{N}(t) = \mathbf{T}'(t)/|\mathbf{T}'(t)|$$

$$\text{Clairaut's thm.: } f_{xy}(a, b) = f_{yx}(a, b)$$

Tan. plane to  $z = f(x, y)$  at  $\langle x_0, y_0, z_0 \rangle$ :

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

$$\text{Grad.: } \nabla f(x, y) = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j}.$$

Dir. deriv. towards  $\mathbf{u}$  at  $\langle x_0, y_0 \rangle$ :

$$D_{\langle a, b \rangle} f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b = \nabla f(x_0, y_0) \cdot \mathbf{u}.$$

Max of  $D_{\mathbf{u}} f(\mathbf{x}) = |\nabla f(\mathbf{x})|$ . Tan. plane of  $f$  at  $\mathbf{p}$ :

$$0 = f_x(\mathbf{p})(x - p_x) + f_y(\mathbf{p})(y - p_y) + f_z(\mathbf{p})(z - p_z).$$

If  $f$  has loc. extrem. at  $\mathbf{p}$ , then  $f_x(\mathbf{p}) = 0$  (&  $f_y$ , etc). If so, let

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2.$$

$D = 0$ : no information.  $D < 0$ : saddle pt.  $D > 0$ :  $f_{xx}(\mathbf{p}) > 0 \implies$  loc. min;  $f_{xx}(\mathbf{p}) < 0 \implies$  loc. max. ( $D$  is the **Hessian mat.**)

Set of possible abs. min and max vals of  $f$  in reg.  $D$ :  $f$  at critical pts. and extreme vals. on the boundary of  $D$ .

Lagrange mults.: extreme vals of  $f(\mathbf{p})$  when  $g(\mathbf{p}) = k$ . Find all  $\mathbf{x}, \lambda$  s.t.

$$\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x}), \quad g(\mathbf{x}) = k.$$

i.e.  $f_x = \lambda g_x$ , etc.

$$\iint f(r \cos \theta, r \sin \theta) r dr d\theta.$$

$$A = \iint_D \left( \sqrt{f_x(x, y)^2 + f_y(x, y)^2 + 1} \right) dA.$$

Line int.s

$$\int_C f(x, y) ds =$$

$$\int_a^b f(x(t), y(t)) \sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2} dt$$

If  $C$  is a smooth curve given by  $\mathbf{r}(t)$  from  $a \leq t \leq b$ ,

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

Spherical coords:  $x = \rho \sin \phi \cos \theta$

$$y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi$$

$$\text{curl } \mathbf{F} =$$

$$\left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle.$$

$$\mathbf{F} = \langle P, Q, R \rangle, \quad \text{curl } \mathbf{F} = \nabla \times \mathbf{F}$$

$$\mathbf{F} \text{ "conservative" } \implies \exists f, \mathbf{F} = \nabla f.$$

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

$$\text{curl}(\nabla f) = \mathbf{0}, \quad \text{div curl } \mathbf{F} = 0$$

If  $C$  is a positively-oriented (ccw) closed curve,  $D$  is bounded by  $C$ , and  $\mathbf{n}$  represents the normal,

$$\oint_C \mathbf{F} \cdot \mathbf{n} ds = \iint_D \text{div } \mathbf{F}(x, y) dA.$$

Common derivs:  $f(g(x)) \rightarrow g'(x)f'(g(x))$ ,  
 $b^x \rightarrow b^x \ln b$ ,  $f^{-1}(x) \rightarrow 1/f'(f^{-1}(x))$ ,  
 $\ln x \rightarrow 1/x$ ,  $\sin x \rightarrow \cos x$ ,  $\cos x \rightarrow -\sin x$ ,  
 $\tan x \rightarrow \sec^2 x$ ,  $\sin^{-1} x \rightarrow 1/\sqrt{1-x^2}$ ,  
 $\cos^{-1} x \rightarrow -(\sin^{-1} x)'$  (etc.),  
 $\tan^{-1} x \rightarrow 1/(1+x^2)$ ,  
 $\sec^{-1} x \rightarrow 1/(|x|\sqrt{x^2-1})$ .

Common ints (don't forget +C):

$$x^n \rightarrow \frac{x^{n+1}}{n+1} + C \quad \text{when } n \neq -1$$

$$1/x \rightarrow \ln|x|$$

$$\tan x \rightarrow -\ln|\cos x|$$

$$\int uv' dx = uv - \int u'v dx \quad (\text{Int. by parts})$$

$$\int u dv = uv - \int v du$$

$$\int_{g(a)}^{g(b)} f(u) du = \int_a^b f(g(x))g'(x) dx \quad \text{u-substitution.}$$

E.x. in  $\int 2x \cos x^2 dx$ , let  $u = x^2$ , find

$du/dx = 2x \implies du = 2x dx$ , subs.

$\int \cos u du = \sin u + C = \sin x^2 + C$ .

$$\iint_{\mathcal{R}} f(x, y) dA = \int_{\alpha}^{\beta} \int_a^b f(r \cos \theta, r \sin \theta) r dr d\theta$$

- Integrand contains  $a^2 - x^2$ , let  $x = a \sin \theta$  and use  $1 - \sin^2 \theta = \cos^2 \theta$ .
- $a^2 + x^2$ , let  $x = a \tan \theta$ , use  $1 + \tan^2 \theta = \sec^2 \theta$ .

- $x^2 - a^2$ , let  $x = a \sec \theta$ , use  $\sec^2 \theta - 1 = \tan^2 \theta$ .

$$\lim_{x \rightarrow 0} \sin x/x = 1$$

$$\lim_{x \rightarrow 0} (1 - \cos x)/x = 0$$

$$\lim_{x \rightarrow \infty} x \sin(1/x) = 1$$

$$\lim_{x \rightarrow 0} (1+x)^{1/x} = e$$

$$\lim_{x \rightarrow 0} (e^{ax} - 1)/(bx) = a/b$$

$$\lim_{x \rightarrow 0^+} x^x = 1$$

$$\lim_{x \rightarrow 0^+} x^{-n} = \infty$$

$$\text{For } 0/0 \text{ or } \pm\infty/\infty, \quad \lim_{x \rightarrow c} f(x)/g(x) = \lim_{x \rightarrow c} f'(x)/g'(x)$$

$$\text{For } g(x) \text{ cont. at } L, \quad \lim_{x \rightarrow c} f(x) = L \implies \lim_{x \rightarrow c} g(L)$$